

UNOTE 2007 EXAM PROBLEMS

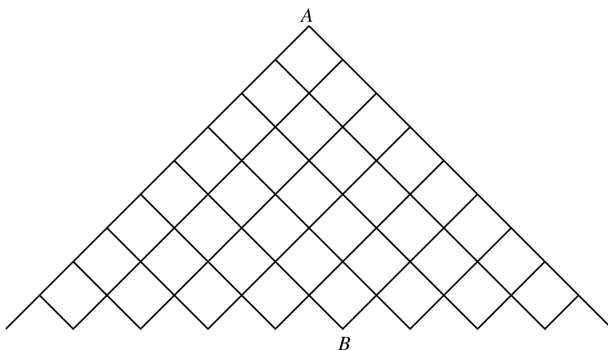
- (1) In a large urn there are 2007 orange balls and 2008 yellow balls. Next to the urn is a large pile of yellow balls. The following procedure is performed repeatedly.

Two balls are chosen from the urn:

- (a) If both are yellow, one is put back, the other thrown away;
- (b) If both are orange, they are both thrown away and a yellow ball from the pile is put into the urn;
- (c) If they are of different colors, the orange one is put back into the urn and the yellow one is thrown away.

What is the color of the last ball in the urn?

- (2) In the graph below, how many paths that never go up connect node A to node B ? (Paths must follow edges indicated.)



- (3) For all x in the interval $0 \leq x \leq \pi$, prove that

$$|\sin(nx)| \leq n \sin x, \quad n \text{ a nonnegative integer.}$$

- (4) Find the volume of the octahedron inscribed in a sphere of radius 1.
- (5) Show that $n^4 - 20n^2 + 4$ is composite when n is any integer.
- (6) A rectangle is inscribed in a sector of a circle of radius 1 as shown in the figure below. The central angle of the sector is a given angle φ , with $0 < \varphi \leq \pi/2$. Show that the maximum possible area for the rectangle is

$$\frac{1 - \cos \varphi}{2 \sin \varphi}.$$

