SAMPLE UNOTE PROBLEMS

(Note: No calculators are allowed in any UNO contest.)

(1) In how many ways can 4 lions and 8 tigers be set in a row, if no two lions can be together? (All 12 beasts are distinguishable from each other!)

(2) Prove that
$$x^4 + 2x^2 + 2x + 2$$
is not the product of two polynomials
$$x^2 + ax + b \qquad \text{and} \qquad x^2 + cx + d$$
in which a, b, c, d are integers.

(3) How many rectangles (of all sizes) are to be found on an 8×8 chessboard? How many of them are squares (of any size)?

(4) Let n be a given positive number. How many pairs (x, y) of positive integers solve the equation
$$\frac{xy}{x+y} = n?$$

(5) Let $a > 0$ be a real number. Show that, among the positive numbers
$$a, 2a, \ldots, (n-1)a,$$
there is one that differs from an integer by at most $1/n$.

(6) Find the number of pairs of integers x, y between 1 and 1000 such that $x^2 + y^2$ is divisible by 49.

(7) Given any $n + 1$ integers between 1 and $2n$, show that one of them is divisible by another.

(8) Given 19 lattice points on the plane (lattice points are those having integer coordinates) show that some three of them are the vertices of a triangle whose center of gravity is a lattice point.

(9) Joe and Jill have agreed to meet at a cafeteria between 6 & 7 PM. However, each of them will just randomly show up during this time, and wait for the other for only 10 minutes. What is the probability that they will actually meet?

(10) Let C be the unit circle $x^2+y^2 = 1$. A point p is chosen randomly on the circumference of C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x- and y-axes with diagonal pq. What is the probability that no point of R lies outside of C?

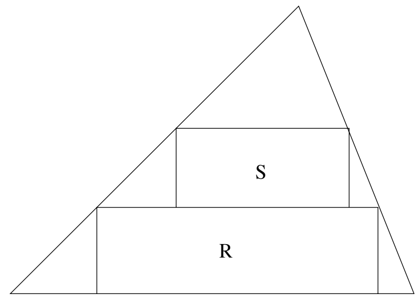
(11) Prove that
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{1999} - \frac{1}{2000} = \frac{1}{1001} + \frac{1}{1002} + \cdots + \frac{1}{2000},$$
where the signs are alternating on the left side, but are all alike on the right side.

(12) Say that a positive integer n is a sum of consecutive integers if there exist positive integers m and k such that $n = m + (m+1) + \cdots + (m+k)$. Prove that n is so expressible if and only if it is not a power of 2.

(13) Let the integer $n > 1$ have the prime factorization
$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$$
where the primes p_1, p_2, $\ldots$, p_k are distinct.
  (a) Find a formula for the number of positive divisors of n.
  (b) Find the sum of all the positive divisors of n.

(14) Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that
(a) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and
(b) $A_1 \cap A_2 \cap A_3 = \emptyset$,
where $\emptyset$ denotes the empty set. Express the answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, and d are nonnegative integers.

(15) Let T be an acute triangle. Inscribe a pair R, S of rectangles in T as shown:



Let $A(X)$ denote the area of polygon X. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above.

(16) Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$